For very special loadings it is possible, however, that more than one value of a critical force is physically meaningful This was brought out in a recent investigation of the present writers,[2] in which it was found that a two-degree-of-freedom system may possess multiple stable and unstable ranges of the load The number of physically meaningful critical values of the load has to be odd, because, before a higher critical value is reached, the system, under gradually increasing loads, has to become stable again

Even such type of loading, however, could hardly be held responsible for the scatter of shell buckling loads As pointed out by Bolotin,[3] not a single experiment has ever been carried out in which buckling would have been produced by a nonconservative static force The fact of the matter is that such forces are quite easily introduced into the analytical treatment of a model by means of arrows, but their realizability in a test presents great difficulties Niedenfuhr expects that fluid pressure forces acting on a shell are nonconservative, but this would be true only if it were possible to exert this pressure over a limited area of the shell surface, without applying any other forces, as discussed more fully in Ref 3

Two further aspects of dynamic buckling under nonconservative static forces render its usefulness even more questionable for the purpose of comparing analytical and experimental buckling results The first concerns the peculiar role of damping played in such systems Even vanishing damping, in general, lowers the buckling load[3] and makes it depend in a two-degree-of-freedom system on the ratio of damping of constants for each generalized coordinate Thus, if the loads were nonconservative, damping should have been included in the analysis

The second aspect is the following In the absence of damping, the dynamic buckling load is characterized by two natural frequencies approaching each other as the loading increases and coinciding at the critical value of the loading It is known, however, from the theory of stability of motion that, whenever two frequencies coincide, the usual stability criteria of Routh-Hurwitz might lead to erroneous results, and then a nonlinear analysis has to be carried out Thus, the buckling loads determined from a "small" vibration analysis might be quite inaccurate, and no good correlation with experiments, even if it were possible to carry them out, is to be expected

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Reply by Author to G Herrmann and R W Bungay

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Having studied the writers' arguments in the preceding comment, the author remains unconvinced of their validity The point of the example in the original note is that, even though the parameters of a system have become such as to render it susceptible to dynamic failure, the system may still be statically stable The dynamic modes of deformation may then provide a mechanism for the system to pass from one branch to another of the static equilibrium locus by paths that do not lie on this locus and that may bypass static critical loads As to the scatter, firstly, the fact that a real system may be susceptible to dynamic failure does not mean that it must fail, merely that it will fail if it is subjected to the proper disturbance The load level at which this disturbance is introduced is generally an indeterminate quantity Secondly, the precise load level at which a system becomes susceptible to dynamic failure can in a real system be affected by assembly details, particularly by the amount of dry friction present It is clear, for instance, that dry friction in the hinge in the middle of the compound bar of the original example will profoundly influence the Beck load for the system It is difficult to judge the appropriateness of the writers' Ref 2 since it has not, as of this writing, appeared in print

The author believes that Bolotin's statement (Ref 3 of preceding comment) here is beside the point The unsteady hydrodynamic forces associated with large local deformations of the shell are surely not completely conservative The only question is the effect of the nonconservative components of these forces Their control or elimination in a test admittedly presents great difficulties, but their realization is almost unavoidable

The term "dynamic buckling" here is perhaps an unfortunate one in that it does not illuminate the mechanism of the failure which is precisely the same as that of subsonic wing flutter Indeed, "flutter buckling" would be a much more descriptive term Making use of the analogy thus introduced, one can envision how the introduction of damping might affect the buckling load either downward by increasing the coupling between modes or upward by adding to the effective stiffness of the system

It is of course true that a nonlinear analysis is necessary to determine the buckled configuration of a system All that a linear analysis can do is to determine the critical loads (and even these may even be affected by the choice of coordinates, as is pointed out in Ref 1) In this connection, however, the following theorem due to Lyapunov gives the engineer some faith in the efficacy of linear analysis

Let $F_i(x_1,x_2, \quad)$ be functions of the dynamical variables which are of at least second degree in the x's, and consider the so-called linearizable system given by $\dot{x}_i = a_{ij}x_j + F_i(x_1,x_2, \quad)$, where the a_{ij} are constants Then, according to Lyapunov, if the linearized system $\dot{x}_i = a_{ij}x_j$ is stable (in the sense of having the real parts of each of its characteristic numbers be negative), the original system is stable no matter what the functions F_i may be

Reference

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Calculation of Gravitational Force Components

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The components of the earth gravitational force are computed as the gradient of an assumed geopotential function When the function is simple, perhaps involving only a few of the zonal harmonics, it and its gradient may reasonably be stated directly in terms of rectangular position co-

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ordinates Such has been a common approach, but it is one that leads to exceedingly cumbersome formulas when applied to a general potential function A preferred method in this case is to form the gradient in a local horizontal coordinate system and then obtain the desired representation of the gradient by a coordinate transformation The details of the method and some comments upon it follow

Method

The general geopotential function U may be written† as

$$U = \frac{\mu}{r}\left[1 - \sum_{n=2}^{n_1} J_n \left(\frac{a_e}{r}\right)^n P_n(\sin\varphi) + \sum_{n=2}^{n_2}\sum_{m=1}^{n} J_{nm}\left(\frac{a_e}{r}\right)^n P_n^m(\sin\varphi)\cos m(\lambda - \lambda_{nm})\right]$$

where

μ	=	product GM of the Newtonian gravitational constant and the mass of the earth
r,φ,λ	=	geocentric distance, geocentric latitude, and (east) longitude of a point
a	=	mean equatorial radius of the earth
J_n, J_{nm}	=	numerical coefficients
P_n	=	Legendre polynomial of the first kind of degree n
P_n^m	=	Legendre associated function of the first kind
λ_{nm}	=	longitudes associated with the J_{nm}

In the local horizontal coordinate system, in which the coordinate axes are directed up (along the radius vector), east, and north (see Fig 1), the force components are

$$g_U = \frac{\partial U}{\partial r} = -\frac{\mu}{r^2}\left[1 - \sum_{n=2}^{n_1}(n+1)J_n\left(\frac{a}{r}\right)^n P_n(\sin\varphi) + \sum_{n=2}^{n_2}\sum_{m=1}^{n}(n+1)J_{nm}\left(\frac{a_e}{r}\right)^n P_n^m(\sin\varphi)\cos m(\lambda-\lambda_{nm})\right]$$

$$g_E = \frac{1}{r\cos\varphi}\frac{\partial U}{\partial\lambda} = -\frac{\mu}{r^2}\sum_{n=2}^{n_2}\sum_{m=1}^{n} mJ_{nm}\left(\frac{a}{r}\right)^n \frac{P_n^m(\sin\varphi)}{\cos\varphi}\sin m(\lambda-\lambda_{nm})$$

$$g_N = \frac{1}{r}\frac{\partial U}{\partial\varphi} = -\frac{\mu}{r^2}\left[\sum_{n=2}^{n_1} J_n\left(\frac{a}{r}\right)^n P_n'(\sin\varphi)\cos\varphi - \sum_{n=2}^{n_2}\sum_{m=1}^{n} J_{nm}\left(\frac{a_e}{r}\right)^n P_n^{m\prime}(\sin\varphi)\cos\varphi\cos m(\lambda-\lambda_{nm})\right]$$

The Legendre functions and their derivatives are computed from the recursion formulas

$$P_n(\sin\varphi) = \frac{-(n-1)P_{n-2}(\sin\varphi) + (2n-1)\sin\varphi P_{n-1}(\sin\varphi)}{n}$$

$$P_n'(\sin\varphi) = \sin\varphi P_{n-1}'(\sin\varphi) + nP_{n-1}(\sin\varphi)$$

$$\frac{P_n^m(\sin\varphi)}{\cos\varphi} = \frac{-(n+m-1)[P_{n-2}^m(\sin\varphi)/\cos\varphi] + (2n-1)\sin\varphi[P_{n-1}^m(\sin\varphi)/\cos\varphi]}{n-m}$$

$$\frac{P_m^m(\sin\varphi)}{\cos\varphi} = 1\ 3 \qquad (2m-1)(\cos\varphi)^{m-1}$$

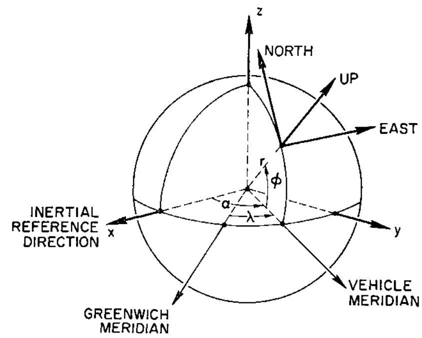

Fig 1 Coordinate systems

$$P_n^{m\prime}(\sin\varphi)\cos\varphi = (n+1)\sin\varphi\frac{P_n^m(\sin\varphi)}{\cos\varphi} - (n-m+1)\frac{P_{n+1}^m(\sin\varphi)}{\cos\varphi}$$

with the initial values

$$P_0(\sin\varphi) = P_1'(\sin\varphi) = 1$$

$$P_1(\sin\varphi) = \sin\varphi$$

$$P_{m-1}^m(\sin\varphi)/\cos\varphi = 0$$

The use of the quotient $P_n^m(\sin\varphi)/\cos\varphi$ avoids numerical difficulties at high latitudes in the equation for g_E

The components of the force vector in an equatorial inertial coordinate system with the x axis as the reference direction are

$$\begin{pmatrix} g_x \\ g_y \\ g \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\alpha & -\sin\alpha & -\sin\varphi\cos\alpha \\ \cos\varphi\sin\alpha & \cos\alpha & -\sin\varphi\sin\alpha \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix}\begin{pmatrix} g_U \\ g_E \\ g_N \end{pmatrix}$$

where α is the right ascension (inertial longitude) of the vehicle

Comments

The merits of the method‡ (which include the comparative simplicity of formulation, programming, and checking, and the ease with which higher-order terms may be incorporated into a computer program) all derive from the natural coordinate system in which the gradient is formed

Sines and cosines (rather than the angles themselves) should be the independent variables of computation They are usually directly available [$\sin\varphi = z/(x^2+y^2+z^2)^{1/2}$ and $\cos\alpha = x/(x^2+y^2)^{1/2}$, for example], and their use avoids both trigonometric evaluations and possible loss of precision in such formulas as $\cos\varphi = (1-\sin^2\varphi)^{1/2}$

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† This form is an extension of the axially symmetric form first used in 1958 by Merson and King-Hele,[1] by Vinti[2] in 1959, and recommended by Brouwer[3] and by Hagihara[4] to the International Astronomical Union

‡ An IBM 7090 subroutine (ASC GRAV) embodying this method is available from SHARE or the author